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# Fermions and Expanding Universe

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The coupled *Dirac–Einstein* equations for a homogeneous isotropic space-time forbid a *closed* universe but lead to the standard cosmological model for a *flat* universe. Therefore only the *open* universe is left as a nontrivial situation. There some of the desired cosmological effects emerge in a natural way: *inflation*, *creation ex nihilo*, etc.

## 1. INTRODUCTION

The well-known shortcomings of the standard cosmological model are generally held to be successfully overcome by the phenomenon of inflation (Blau and Guth, 1987; Turner, 1986). For the working of that inflationary mechanism, there must exist a weakly coupled scalar field which is initially in the false vacuum and therefore drives the inflation until the right vacuum value is attained. Although this paradigm has recently evoked some controversy [see the discussion of S. W. Hawking in Lightman and Brawyer, 1990; Penrose, 1989], the very occurrence of inflation is nevertheless thought to yield a very satisfactory explanation of the universe's past history. Therefore the controversial viewpoints are not so much concerned with its *existence*, but rather with a detailed *foundation* of inflation, especially with the question of where the scalar field comes from and why it undergoes such a peculiar phase transition from the false into the right vacuum, as is suggested by the Higgs mechanism.

In such an ambiguous situation, it may be favorable to reconsider alternative possibilities for the inflationary mechanism. These can roughly be subdivided into two categories: (i) modifying Einstein's gravitation theory or (ii) resorting to exotic equations of state for matter ($\rightsquigarrow$ negative

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pressure). An example of the first category (i) has recently been proposed (Mattes and Sorg, 1991) and in the present paper we consider an alternative of the second kind (ii). Here, the matter content of the primeval universe is not constituted by some *scalar* field (of questionable origin), but by a Dirac *spinor* field, whose existence in nature is beyond any doubt.

Our results are the following: The Dirac–Einstein equations admit as a nontrivial case only the *open* universe. Here, there exist two types of solutions. The first type yields a universe which starts with zero radius but is "nonsingular" at the time $t=0$ of its creation, i.e., the matter density $\rho(t=0)$ is finite and consequently the total mass is zero. Shortly after the universe's creation, the Dirac pressure is negative and therefore drives inflation during a certain time period lasting until the extension of the universe roughly reaches the magnitude of the Compton wavelength of the Dirac particle. After this inflationary period, the cosmic evolution rapidly approaches the situation predicted by the standard model. The mass-energy content of a comoving 3-volume is essentially generated during the inflationary phase and becomes constant during the subsequent standard phase. This feature is shared also by the second type, the bounce solutions; however, the comoving mass can adopt here two different, asymptotically constant values before and after the bounce.

## 2. DIRAC–EINSTEIN EQUATIONS

The point of departure is the assumption that the matter content of the primordial universe has been dominantly constituted by a Dirac spinor field $\psi$, whose energy-momentum content $\mathbf{T}[\psi]$ enters the Einstein field equations in the usual way,

$$R_{\mu\nu} - \frac{1}{2} RG_{\mu\nu} = 8\pi \left(\frac{L_p^2}{\hbar c}\right) T_{\mu\nu} \tag{1}$$

where $L_p$ is the Planck length. Here we restrict ourselves to a homogeneous isotropic universe, the metric tensor $\mathbf{G}$ of which is conveniently split up into two projectors according to

$$G_{\mu\nu} = \mathscr{B}_{\mu\nu} + b_\mu b_\nu \tag{2}$$

The timelike part is formed by the unit vector $\mathbf{b}$,

$$b^\mu b_\mu = 1 \tag{3}$$

while the spacelike part $\mathscr{B}$ satisfies

$$\mathscr{B}_{\mu\nu} b^\nu = 0 \tag{4a}$$

$$\mathscr{B}_{\mu\nu} \mathscr{B}^\nu{}_\lambda = \mathscr{B}_{\mu\lambda} \tag{4b}$$

For such a special universe, the Einstein tensor **E** on the left of equation (1) acquires the following shape (Sorg, 1992):

$$\begin{aligned} E_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} \\ &= \left(2\frac{\ddot{\mathscr{R}}}{\mathscr{R}} + H^2 - \frac{\sigma}{\mathscr{R}^2}\right)\mathscr{B}_{\mu\nu} + 3\left(H^2 - \frac{\sigma}{\mathscr{R}^2}\right) b_\mu b_\nu \end{aligned} \tag{5}$$

where $\mathscr{R}$ is the radius of the universe and a dot denotes the derivative with respect to cosmic time $\theta$. Thus the Hubble rate $H$ is given by

$$H \equiv \frac{\dot{\mathscr{R}}}{\mathscr{R}} \tag{6}$$

Moreover, the "foliation index" $\sigma$ in equation (5) specifies the topology of the time slices $\theta = \text{const}$ in the usual way: $\sigma = +1$ stands for the open universe, and $\sigma = 0$ $(-1)$ denotes the flat (closed) cases, respectively.

An important consequence of the homogenity and isotropy assumption (5) is now that the Einstein equations (1) force the energy-momentum $\mathbf{T}[\psi]$ of the Dirac spinor field $\psi$ into a very special shape, namely

$$T_{\mu\nu} = \mathscr{M} b_\mu b_\nu - \mathscr{P}\mathscr{B}_{\mu\nu} \tag{7}$$

Here the energy density $\mathscr{M}$ and pressure $\mathscr{P}$ are constant over the time slices and depend therefore exclusively upon cosmic time $\theta$. Such a configuration is not quite trivial for a spinor field, because its spin density normally breaks isotropy. Thus we have to look for a method for obtaining in curved space those solutions of the Dirac equation

$$i\hbar\gamma^\mu \mathscr{D}_\mu \psi = Mc\psi \tag{8}$$

which carry an energy-momentum content just of the special kind (7).

## 3. RELATIVISTIC SCHRÖDINGER EQUATIONS

Our method consists in rewriting the Dirac equation (8) in form of a "relativistic Schrödinger equation" (Sorg, 1992)

$$i\hbar c(\mathscr{D}_\mu \psi) = \mathscr{H}_\mu \psi \tag{9}$$

The gauge-covariant derivative $\mathscr{D}_\mu$ is defined here as usual

$$\mathscr{D}_\mu \psi := \partial_\mu \psi + \mathscr{A}_\mu \cdot \psi \tag{10}$$

and involves the connection $\mathscr{A} = \mathscr{A}_\mu \, dx^\mu$

$$\mathscr{A}_\mu = \tfrac{1}{2}\omega_{\alpha\beta\mu}\hat{\Sigma}^{\alpha\beta} \tag{11}$$

as a $\mathscr{SPIN}(1, 3)$-valued 1-form with the corresponding generators $\hat{\Sigma}$ being built up by the ordinary Dirac matrices $\hat{\gamma}^\alpha$ through

$$\hat{\Sigma}^{\alpha\beta} = \tfrac{1}{4}[\hat{\gamma}^\alpha, \hat{\gamma}^\beta] \tag{12}$$

Moreover, the Hamiltonian $\mathscr{H} = \mathscr{H}_\mu \, dx^\mu$ is a $\mathscr{G}l(4, \mathbb{C})$-valued 1-form and transforms homogeneously with respect to a gauge transformation of the wave function $\psi$

$$\psi' = \mathscr{S}^{-1} \cdot \psi \tag{13}$$

i.e.,

$$\mathscr{H}'_\mu = \mathscr{S}^{-1} \cdot \mathscr{H}_\mu \cdot \mathscr{S} \tag{14}$$

It is required to satisfy the "Dirac condition" (Sorg, 1992)

$$\gamma^\mu \cdot \mathscr{H}_\mu = Mc^2 \cdot 1 \tag{15}$$

in order that any solution of the relativistic Schrödinger equation (9) also obeys the Dirac equation (8).

Thus, we can construct the solution $\psi(x)$ of the latter equation in the presence of an external field $\mathscr{F}$ $(= d\mathscr{A} + \mathscr{A} \wedge \mathscr{A})$ by first looking for the Hamiltonian $\mathscr{H}$ and then integrating the corresponding Schrödinger equation (9) for $\psi$. The dynamical equations for the Hamiltonian $\mathscr{H}_\mu$ $(:= \hbar c \mathscr{H}_\mu)$ are easily deduced as (Sorg, 1992)

$$\mathscr{D}_\mu \mathscr{H}_\nu - \mathscr{D}_\nu \mathscr{H}_\mu + i[\mathscr{H}_\mu, \mathscr{H}_\nu] = i\mathscr{F}_{\mu\nu} \tag{16a}$$

$$\mathscr{D}_\mu \mathscr{H}^\mu - i(\mathscr{H}_\mu \cdot \mathscr{H}^\mu) + i\left(\frac{Mc}{\hbar}\right)^2 \cdot 1 = -i\Sigma^{\mu\nu} \cdot \mathscr{F}_{\mu\nu} \tag{16b}$$

Such a Hamiltonian dynamics immediately guarantees conservation laws of the kind

$$\nabla_\mu j^\mu = 0 \tag{17}$$

and

$$\nabla_\mu T^{\mu\nu} = 0 \tag{18}$$

where the densities **j** and **T** are bilinear constructions of the wave function $\psi$ and the corresponding operators, i.e.,

$$j^\mu = \bar{\psi} \cdot v^\mu \cdot \psi \tag{19}$$

and

$$T^{\mu\nu} = \bar{\psi} \cdot \mathscr{T}^{\mu\nu} \cdot \psi \tag{20}$$

For instance, for the Dirac field the "velocity operator" $v$ just coincides with the Dirac matrices in curved space ($v^\mu \Rightarrow \gamma^\mu$), and the energy-momentum operator $\mathscr{T}$ is found as

$$\begin{aligned} \mathscr{T}_{\mu\nu} &= \tfrac{1}{2}(\gamma_\mu \cdot \mathscr{H}_\nu + \bar{\mathscr{H}}_\nu \cdot \gamma_\mu) \\ \bar{\mathscr{H}} &:= \hat{\gamma}^0 \cdot \mathscr{H}^+ \cdot \hat{\gamma}^0 \end{aligned} \tag{21}$$

The advantage of the relativistic Schrödinger approach is now that for solving the Einstein equations (1) one is not forced to actually compute the Dirac wave function $\psi$! Rather, it is sufficient to consider merely those densities which enter equation (1); their dynamical equations are also governed by the Hamiltonian $\mathscr{H}$. Therefore we can restrict ourselves to the determination of $\mathscr{H}$, in place of $\psi$.

## 4. EQUATION OF STATE

To give an example, we return to the homogeneous, isotropic space-time as specified by equations (5) and (7). For such a highly symmetric situation, the Hamiltonian $\mathscr{H}$ may be combined from the generators of $\mathscr{G}l(4, \mathbb{C})$ in a very simple way. This Lie algebra is isomorphic to the Clifford algebra $\mathscr{C}(1, 3)$ and therefore is spanned up by the scalar density operator 1, by its pseudocounterpart $\varepsilon$ $(:= (1/4!)\varepsilon_{\mu\nu\lambda\sigma}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)$, further by the velocity operator $\gamma^\mu$ and its axial counterpart $\tilde{\gamma}^\mu = \varepsilon \cdot \gamma^\mu$, and finally by the $\mathscr{SPIN}(1, 3)$ generators $\Sigma^{\mu\nu}$. Therefore our Hamiltonian ansatz is constructed by these elements in the following manner:

$$\begin{aligned} \mathscr{H}_\mu \Rightarrow {}^{(W)}\mathscr{H}_\mu = \frac{Mc}{4\hbar}\gamma_\mu + i\left(\frac{\sigma}{\mathscr{R}}\sin\chi - H\right)\left(\frac{3}{2}b_\mu \cdot 1 - b^\lambda \Sigma_{\mu\lambda}\right) \\ + \left(\frac{\sigma}{2\mathscr{R}}\cos\chi + \frac{Mc}{4\hbar}\right)(4b_\mu b_\lambda - G_{\mu\lambda})\gamma^\lambda \end{aligned} \tag{22}$$

Observe here that, besides the radius $\mathscr{R}$ and Hubble expansion rate $H$, there emerges only a single additional dynamical variable, namely the angle $\chi$, which is to be considered as a function of cosmic time $\theta$. Moreover, for a free particle in flat space-time ($H = \sigma = 0$) the Hamiltonian ${}^{(W)}\mathscr{H}$ of (22) is simplified into

$$^{(W)}\mathscr{H}_\mu \Rightarrow {}^{(0)}\mathscr{H}_\mu = \frac{Mc}{\hbar}b_\mu(b_\lambda\gamma^\lambda) \tag{23}$$

and implies the well-known plane wave solutions

$$\psi(x) \Rightarrow {}^{(0)}\psi(x) = \psi_{\text{in}} \exp[\mp ik_\mu x^\mu] \tag{24}$$

of the Dirac equation (8), resp. Schrödinger equation (9), provided the initial value $\psi_{\text{in}}$ is an eigenspinor of the Hamiltonian ${}^{(0)}\mathscr{H}$:

$${}^{(0)}\mathscr{H}_\mu \psi_{\text{in}} = \pm k_\mu \psi_{\text{in}} \tag{25}$$

[The eigenspinors $\psi_{\text{in}}$ can easily be determined (Sorg, 1992).] The interesting point with those flat solutions is here that, despite their nonvanishing spin density, their energy-momentum density $\mathbf{T}$ [equations (20), (21)] obeys the homogenity and isotropy requirement (7), i.e.,

$$T_{\mu\nu} \Rightarrow {}^{(0)}T_{\mu\nu} = Mc^2 \varrho b_\mu b_\nu \tag{26}$$

because the scalar density $\varrho$ $(:= \bar{\psi} \cdot \psi)$ is a constant throughout space-time.

The energy-momentum density ${}^{(0)}\mathbf{T}$ of (26) suggests that we interpret —albeit somewhat naively—the scalar $\varrho$ as the particle density, i.e., number of particles per unit 3-volume, with antiparticles counted negative (Misner *et al.*, 1973). Thus the invariant energy density (26) can be thought of as the product of the mass-energy of a single particle ($Mc^2$) times the particle density $\varrho$. Unfortunately, this interpretation evokes a well-known difficulty with the "classical" Dirac field $\psi$, namely its attribution of a negative mass-energy to the antiparticles (the density $\varrho$ is of indefinite sign!). Although this shortcoming of the unquantized Dirac theory may be overcome during the process of second quantization by resorting to the well-known anticommutation relations for the field operators, we can retain here our naive picture for the energy density, because the number ($\mu$) of particles in any comoving 3-volume never changes sign [see equation (40) below].

Fortunately, the pleasant result for that space-time is immediately generalizable to curved space ($H \neq 0$), where one finds by combining equations (20)–(22)

$$(\hbar c)^{-1}\, {}^{(W)}T_{\mu\nu} = 3\varrho \left( \frac{\sigma}{2\mathscr{R}} \cos\chi + \frac{Mc}{3\hbar} \right) b_\mu b_\nu - \varrho \frac{\sigma}{2\mathscr{R}} \cos\chi \mathscr{B}_{\mu\nu} \tag{27}$$

Thus, the equation of state for the Dirac field is deduced from (7) as

$$\mathscr{P} = \beta \mathscr{M} \tag{28}$$

with the thermodynamic coefficient $\beta$ being given by

$$\begin{aligned} \beta &= \sigma \frac{\cos\chi}{3\cos\chi + 2m\mathscr{R}} \\ m &:= \frac{Mc}{\hbar}, \qquad \text{inverse Compton length} \end{aligned} \tag{29}$$

This is a very plausible result because it says that the open ($\sigma = 1$) universe *may* appear radiation-dominated ($\beta = 1/3$) provided its radius $\mathscr{R}$ is much smaller than the Compton length ($m\mathscr{R} \ll 1$), but it always appears matter-dominated ($\beta = 0$) for $m\mathscr{R} \gg 1$. In contrast to this, the flat universe ($\sigma = 0$) is *always* matter-dominated [cf. (26)]. Observe also that the above-mentioned interpretation of the energy density would imply the introduction of an effective particle mass $M_{\text{eff}}$ for the nonflat universes ($\sigma \neq 0$):

$$M_{\text{eff}} = M\left(1 + \frac{3}{2}\sigma\frac{\cos\chi}{m\mathscr{R}}\right) \tag{30}$$

Again this result has a certain plausibility, because one will expect that the proper mass $M$ of a particle is not fully established until the radius of the universe is appreciably greater than its Compton wavelength.

## 5. EXPANSION DYNAMICS

After the energy-momentum content $\mathbf{T}$ of the Dirac wave field is known, it can be used in order to deduce the equation of motion for the radius $\mathscr{R}$ from the Einstein equations (1). Thus, combining both equations (5) and (27) in the required manner readily yields

$$\begin{aligned} \ddot{r} &= -\frac{4\pi}{3}\Lambda^2\frac{\mu}{r^2}\left(1 + 3\sigma\frac{\cos\chi}{r}\right) \\ \mu &:= \varrho R^3 \end{aligned} \tag{31}$$

Here, the radius $\mathscr{R}$ has been rescaled by the Compton length ($r := m\mathscr{R}$), as well as cosmic time $\theta$ ($t := m\theta$) and the Planck length $L_p$ ($\Lambda := mL_p$). Moreover, the density $\varrho$ has been eliminated in favor of the total number $\mu$ of particles in a comoving 3-cell. Besides the dynamical equation for the radius $r$, (31), the Einstein equations also yield the so-called initial-value equation

$$\dot{r}^2 = \sigma + 8\pi\Lambda^2\frac{\mu}{r}\left(\frac{\sigma}{2r}\cos\chi + \frac{1}{3}\right) \tag{32}$$

which is a kind of first integral of the expansion dynamics and may be used to correlate the initial values for its numerical integration.

Obviously, the dynamical equation (31) is not closed and therefore must be complemented by the equations of motion for the angle $\chi$ and density $\varrho$ (resp., particle number $\mu$). The density equation reads quite generally by virtue of the Schrödinger equation (9)

$$\partial_\mu \varrho = i\bar{\psi} \cdot (\bar{\mathscr{H}}_\mu - \mathscr{H}_\mu) \cdot \psi \tag{33}$$

which yields by use of the Hamiltonian ${}^{(W)}\mathscr{H}$ of (22)

$$\dot{\varrho} + 3H\varrho = 3\,\frac{\sigma}{\mathscr{R}}\,\varrho \sin \chi \tag{34}$$

or, with reference to the particle number $\mu$,

$$\dot{\mu} = 3\,\frac{\sigma}{r}\,\mu \sin \chi \tag{35}$$

Finally, for obtaining the last equation of motion we have to specify the Hamiltonian dynamics (16a), (16b) for the ansatz (22), which yields for the angular variable $\chi$

$$\dot{\chi} = \sigma\left(2 + 3\,\frac{\cos \chi}{r}\right) \tag{36a}$$

$$\sigma \neq -1 \tag{36b}$$

The important finding is here that the *closed universe* ($\sigma = -1$) *is forbidden.* However, this negative result is a consequence of the special Hamiltonian ansatz ${}^{(W)}\mathscr{H}$ of (22), and it is not quite sure whether a more general solution of the Hamiltonian dynamics (16a), (16b) would allow also a closed universe. In any case, for the nonclosed topology ($\sigma = 0, 1$) the complete dynamical system consists of the Einstein equation (31), the number dynamics (35), and the angular dynamics (36a). Though the flat case ($\sigma = 0$) is somewhat trivial here, we nevertheless take a glimpse at it because it is a kind of limiting case for the open universe when the radius $\mathscr{R}$ tends to infinity ($\mathscr{R} \to \infty$).

For a flat foliation of space-time ($\sigma = 0$), the constancy of particle number $\mu$ ($\Rightarrow \mu_*$) follows immediately from equation (35) and therefore the density $\varrho$ becomes

$$\begin{gathered} \varrho \Rightarrow {}^{(f)}\varrho = \frac{\mu_*}{\mathscr{R}^3} \\ \mu_* = \text{const} \end{gathered} \tag{37}$$

Similarly, one demands in agreement with equations (36a), (36b) for the angle $\chi$

$$\chi \Rightarrow {}^{(f)}\chi \equiv 0 \tag{38}$$

and then one concludes from either of the equations (31) or (32) for the radius $r$

$$r(t) = (6\pi\Lambda^2\mu_*)^{1/3} t^{2/3} \tag{39}$$

which is a well-known result in standard cosmology [*Einstein–deSitter universe* (Rindler, 1977)].

Obviously, these flat-space results are important also for the open universe ($\sigma = 1$). The reason is that the foliation index $\sigma$ occurs in the dynamical system (31), (32), (35), and (36a) frequently in connection with the inverse radius $r$, so that the flatness assumption ($\sigma = 0$) is equivalent to the large-size approximation $r \to \infty$. Thus, the law of constancy of mass $\mu_*$, (37), in a flat universe is weakened into [cf. (35)]

$$\mu = \mu_* \exp\left[-3\int_\theta^\infty \frac{\sin\chi}{\mathscr{R}}\,d\theta\right] \tag{40}$$

for the open case ($\sigma = 1$), yielding the constancy of particle number only in the limit $\theta \to \infty$ where $\mathscr{R} \to \infty$. Similarly, in this limit the radius $\mathscr{R}$ is found from (31), (32) as $\mathscr{R} \sim \theta$ in place of (39), whereas the angle $\chi$ finally increases linearly with time: $\chi \sim 2t$ ($= 2m\theta$). Consequently, the long-time behavior of the particle number $\mu$ in (40) will be characterized by fluctuations on the Compton length and time scale.

Besides the comparison between the flat and open universes for the present model, it is also instructive to consider the open case of the standard matter-dominated model. Clearly, the conservation of particle number (37) is also valid in this latter case and the corresponding expansion obeys the following law (Rindler, 1977):

$$\frac{t}{(8\pi/3)\Lambda^2\mu_*} = \left[\frac{r}{(8\pi/3)\Lambda^2\mu_*} + \left(\frac{r}{(8\pi/3)\Lambda^2\mu_*}\right)^2\right]^{1/2} - \sinh^{-1}\left[\frac{r}{(8\pi/3)\Lambda^2\mu_*}\right]^{1/2} \tag{41}$$

Since the power law (39) holds also for the present situation (41) in the limit $t \to 0$, all the standard models yield a singular physics at the moment of creation ($t = 0$), which makes them highly unreliable. Against that, one is easily convinced that there exists a nonsingular solution for the present Dirac–Einstein model (31), (32), (35), (36a) which for $t \to 0$ looks as follows:

$$r(t) = t + \mathcal{O}(t^3) \tag{42a}$$

$$\mu(t) = \mu_c t^3 + \mathcal{O}(t^5) \tag{42b}$$

$$\chi(t) = \frac{\pi}{2} + \frac{1}{2}t + \mathcal{O}(t^2) \tag{42c}$$

The remarkable property of this solution is that the particle density $\varrho$ adopts some constant value $\varrho_c$ ($:=\mu_c m^3$) at the moment of creation ($t=0$), so that the particle number $\mu$ vanishes in the first instant. Consequently, the ultimate number $\mu_*$ in (40) of particles must be generated during the subsequent evolution of the universe ($\rightsquigarrow$ creation *ex nihilo*). Observe also that the solution (42a)–(42c) fixes the initial mass $M_{\text{eff}}$ of (30) as a quarter of its ultimate value $M$. We will follow now the further course of events in such a peculiar universe.

## 6. NONSINGULAR UNIVERSE

Among the various problems inherent in the standard cosmological model, perhaps its most acute difficulty is the singularity problem. Indeed, it is hard to imagine how all the mass content ($Mc^2\mu_*$) of the present-day universe could have been concentrated initially ($t \to 0$) in a mathematical point ($\mathscr{R} \to 0$) so that the density $\varrho$ of (37) became infinite. Moreover, it remains unclear why the universe has such a large extension as observed today, because if it came into being with an initial size of roughly the Planck length $L_p$ ($\mathscr{R}_{\text{in}} \approx L_p \sim 10^{-33}$ cm) and was filled with matter of roughly the Planck density ($M\varrho \sim 10^{94}$ g/cm$^3$), then it would have recollapsed within a few Planck times ($\sim 10^{-44}$ sec) according to the Einstein equation of motion

$$\ddot{\mathscr{R}} = -\frac{4\pi}{3}\frac{L_P^2}{\hbar c}(\mathscr{M} + 3\mathscr{P})\mathscr{R} \tag{43}$$

Only if the primordial matter is in an exotic state where it develops negative pressure [e.g., $\beta = -1$; cf. (28)] is it possible to blow up the universe rapidly enough ($\ddot{\mathscr{R}} > 0$) in order to generate the initial outward push for the subsequent standard phase.

Now it is instructive to see in detail how both problems may be overcome by the Dirac–Einstein system (42a)–(42c).

### 6.1. Inflation

First observe that the initial expansion ($r \sim t$) according to (42a) is a consequence of the fact that the initial particle density $\mu$ of (42b) vanishes for $t \to 0$ more rapidly than the radius $r$ itself, so that the initial-value equation (32) admits no other possibility than linear growth for an open universe ($\sigma = 1$). Incidentally, this is the same law of expansion as for long time $t \to \infty$ (Fig. 1). However, this linearity is rapidly abandoned in favor of an exponential growth, which follows from the fact that the coefficient

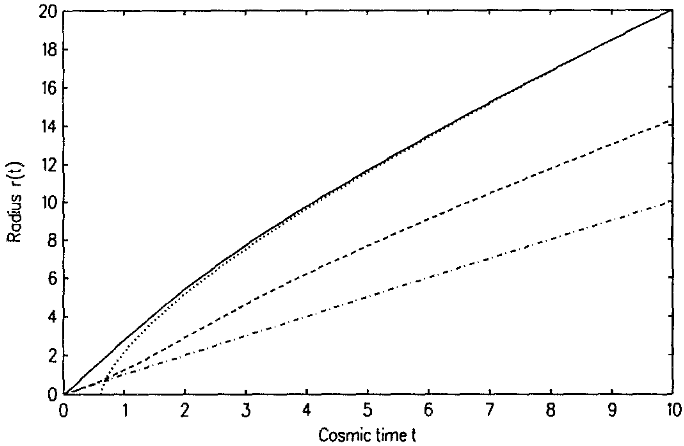


**Fig. 1.** Primeval inflation. The expansion starts with a nonsingular inflationary phase according to (46), but inflation ($\ddot{r} > 0$) is possible only as long as the extension of the universe does not exceed roughly the Compton length $m^{-1}$ of the particle. Beyond this length ($r \gtrsim 1$), the expansion rapidly approaches that of the standard model (41) (dotted curve), which is singular for $t \to 0$ ($r \sim t^{2/3}$) and behaves as $\dot{r} \sim 1$ for $t \to \infty$. For increasing particle mass $m \equiv \Lambda/L_p$ the inflationary push becomes stronger; $\Lambda = 0$: lower curve ($\rightsquigarrow r \equiv t$), $\Lambda = 1$: middle curve; $\Lambda = 100$: upper curve ($\mu_c = 1$ for all cases).

$\beta$ in (29) approaches the exceptional value $\beta = -1$ (Fig. 2) and thus the energy-momentum ${}^{(W)}\mathrm{T}$ of (27) of the Dirac field acts like a cosmological term,

$$ {}^{(W)}T_{\mu\nu} \Rightarrow \tfrac{1}{4} M c^2 \varrho_c G_{\mu\nu} \tag{44} $$

Consequently, the Einstein equation (43), resp. (31), states for this limiting situation

$$ \begin{aligned} \ddot{r} &= \frac{1}{\tau^2} t + \cdots \\ \tau &:= \left( \frac{2\pi}{3} \Lambda^2 m^{-3} \varrho_c \right)^{-1/2} \end{aligned} \tag{45} $$

with the obvious solution

$$ r = t + \frac{\tau}{6} \left( \frac{t}{\tau} \right)^3 + \cdots \tag{46} $$

Thus, the expansion even becomes accelerated ($\ddot{r} > 0$) and consequently undergoes an inflationary phase, whose duration may be estimated through

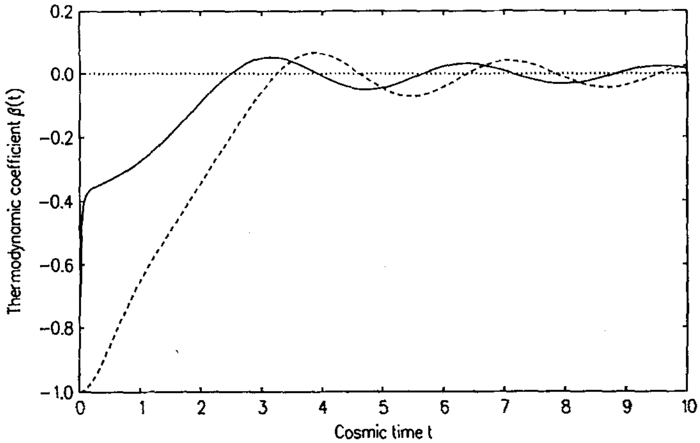


**Fig. 2.** Equation of state for the Dirac field. The coefficient $\beta$ of (29) adopts its exotic value $-1$ in the first instant ($t=0$) and thus signals the inflationary birth of the universe. For large time $t$, $\beta$ adopts the matter-dominance value $\beta=0$. Lower curve: $\Lambda=0$, upper curve: $\Lambda=100$ ($\mu_c=1$ always).

the following argument: Since the equations (31) and (32) do admit only a minimum ($\dot{r}=0$, $\ddot{r}>0$) but not a maximum ($\dot{r}=0$, $\ddot{r}<0$) of the function $r(t)$, the radius $r$ is monotonically increasing with cosmic time $t$ for the solution (42). Therefore, the acceleration $\ddot{r}$ surely remains negative as soon as the radius $r$ exceeds the value 3 ($\rightsquigarrow \mathscr{R}$ is greater than three times the Compton length). Beyond this critical size, the universe looks like the matter-dominated standard model (41), where the normal gravitational pull of the constant matter content breaks the cosmic expansion. As a result we see that inflation can act only as long as the universe is not appreciably greater than the corresponding Compton wavelength, which suggests that we consider inflation as a quantum effect (Fig. 1).

### 6.2. Creation *ex Nihilo*

According to the present picture, the particle number $\mu$ of (42b) vanishes at the moment of creation and therefore also the mass-energy in a comoving 3-cell [cf. (44)]. Thus, the notorious singularity problem of the standard model is avoided and the question of the origin of the universe's mass-energy content is settled trivially: the work-energy theorem for the Dirac tensor ${}^{(W)}T$ of (27),

$$\nabla_\mu T^\mu{}_\nu \equiv 0 \Leftrightarrow d(\mathscr{M}\mathscr{R}^3) = -\mathscr{P}\, d\mathscr{R}^3 \tag{47}$$

simply says that the increase of mass-energy

$$\mathscr{M}\mathscr{R}^3 = M_{\text{eff}} c^2 \mu \tag{48}$$

in a comoving 3-cell of size $\mathscr{R}^3$ is nothing else than the work done by the negative (!) Dirac pressure $\mathscr{P}$ upon this cell:

$$\mathscr{P} = \frac{\sigma}{2\mathscr{R}} \varrho \cos \chi \tag{49}$$

i.e., for the primeval era (42),

$$\mathscr{P} \sim -\tfrac{1}{4}\varrho_c \tag{50}$$

[cf. (44)]. Since the effective mass $M_{\text{eff}}$ per particle is of restricted variability, the main contribution to the increase in the cell energy (48) comes from the generation of new particles, i.e., the rise of the particle number $\mu$ (Fig. 3).

In order to give a rough estimate of this effect, we look for the relationship between the ultimate particle number $\mu_*$ of (40) and the primordial "Compton number" $\mu_c$ of (42b) (i.e., the number of particles in a "Compton volume" of size $m^{-3}$ for $r \to 0$). Here, it may be sufficient for

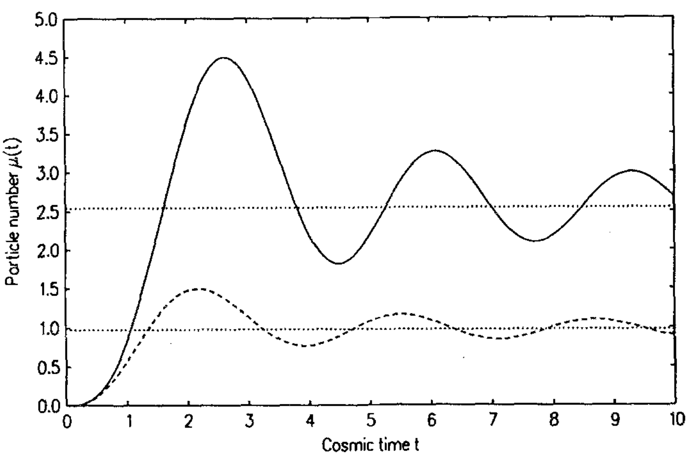


**Fig. 3.** Generation of particles. The universe starts with zero particle number $\mu$, which increases rapidly during the inflationary phase ($t \lesssim 3$) and then is stabilized at a constant value $\mu_*$ corresponding to the standard model. The ultimate value $\mu_*$ of (40) is a rapidly decreasing function $f(\Lambda)$, (57), of the particle mass $m$ ($\equiv \Lambda/L_p$). Upper curve: $\Lambda = 0$; lower curve: $\Lambda = 1$ ($\mu_c = 1$ always).

our qualitative arguments to consider the special case $\Lambda = 0$ (for an electron one has $\Lambda \sim 10^{-25}$; the general dependance of particle number $\mu$ on the parameter $\Lambda$ can be read off from Fig. 3). However, for that simple case $\Lambda = 0$ we conclude from the Einstein equation of motion (31) and the initial-value equation (32) that the radius $r$ is identical to cosmic time $t$ and this simplifies the particle number $\mu$ of (40) into

$$\mu \Rightarrow \mu_* \exp\left[-3\int_t^{\infty} \frac{\sin \chi(t)}{t}\, dt\right] \tag{51}$$

Further, in order to get an approximate value for the integral involved, we use the following estimate for the angle $\chi(t)$:

$$\chi(t) \approx \frac{\pi}{2} + 2t \tag{52}$$

which includes its correct initial value $\chi(0) = \pi/2$ as well as its long-time behavior $\dot{\chi}(t \to \infty) = 2$ (Fig. 4). Thus, for small $t$ we arrive at

$$\int_t^{\infty} \frac{\sin \chi}{t}\, dt = \mathscr{C} - \ln(2t) - \frac{(2t)^2}{2 \cdot 2!} + \frac{(2t)^4}{4 \cdot 4!} \mp \cdots \tag{53}$$

where $\mathscr{C}$ is the Euler number $(= 0.5772\ldots)$. Consequently, the primeval $(t \to 0)$ particle number $\mu$ of (51) is found as

$$\mu(t)|_{\Lambda=0} = 8e^{-3\mathscr{C}}\mu_* t^3 + \mathcal{O}(t^5) \tag{54}$$

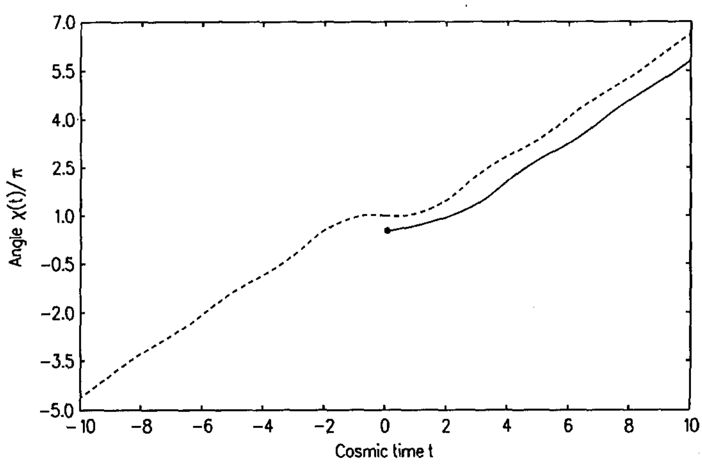


**Fig. 4.** Time dependence of angle $\chi$. For the creation-*ex-nihilo* solution (42) (lower curve), the angle $\chi$ starts with value $\pi/2$ and then approximates $\dot{\chi} = 2$ beyond the inflationary phase; cf. (36a). This behavior induces the number oscillation shown in Fig. 3 and legitimates the estimate (52). Similar oscillations arise for a bounce solution (upper curve).

and the comparison of this result to (42b) yields the desired estimate as

$$\mu_*|_{\Lambda=0} \sim \tfrac{1}{8} e^{3\mathscr{C}} \mu_c \sim 0.7\mu_c \tag{55}$$

Despite its roughness, this approximation confirms the general supposition—suggested also by the numerical results—that the ultimate particle number $\mu_*$ in any comoving 3-cell of size $r(t)^3$ at time $t$ is proportional to $\mu_c$:

$$\mu_*(\Lambda, \mu_c) = f(\Lambda)\mu_c \tag{56}$$

with the unknown function $f(\Lambda)$ still to be determined.

### 6.3. Bounce Solutions

The creation-*ex-nihilo* solution discussed so far is not the only non-singular type admitted by the Dirac–Einstein system (31), (32), (35), (36a) for an open universe $\sigma = +1$. The numerical integrations of that system reveal the existence of *bounce solutions*, where the universe is initially in a state of contraction, which, however, comes to a halt at some minimal radius $r_b$ and then passes over to continual reexpansion (Fig. 5). The

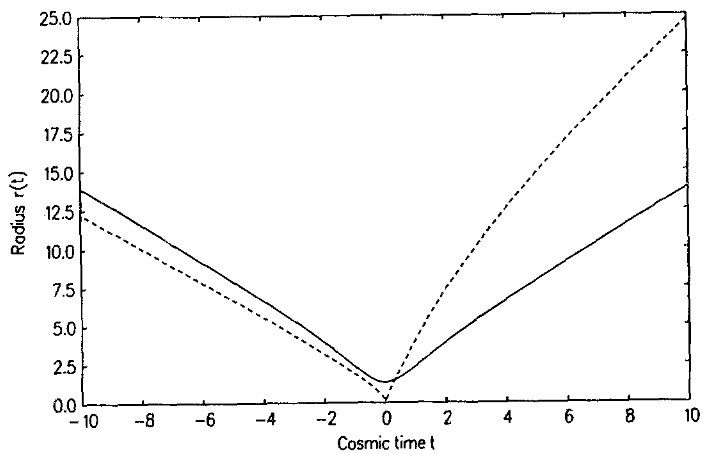


**Fig. 5.** Bounce solutions. The bounce radius $r_b$ cannot exceed the maximal value 3/2 [cf. (57)], where the bounce is symmetric and as soft as possible. For smaller values $r_b$ the bounce acceleration $\ddot{r}_b$ increases and ultimately produces a kick for $r_b \to 0$. Soft bounce: $\chi_b = \pi$. Hard bounce: $\chi_b = 0.55\pi$. ($\Lambda = \mu_b = 1$ always.)

"bounce radius" $r_b$ depends sensitively upon the "bounce angle" $\chi_b$ $(:= \chi|_{rb})$ and is given approximately through

$$\begin{gathered} r_b \sim -\tfrac{3}{2}\cos\chi_b \\ \pi/2 < \chi_b < 3\pi/2 \end{gathered} \tag{57}$$

provided the "bounce number" $\mu_b$ is taken to be large ($\mu_b \gg 1$).

For general initial conditions, the bounce solution will not be time-symmetric with respect to the bounce instant ($\dot{r} = 0$). This becomes especially evident by considering the particle number $\mu$, which is constant sufficiently far away from the bounce (Fig. 6), but this constant is not necessarily the same before and after the bounce. If the bounce angle $\chi_b$ is in the interval $\pi/2 < \chi_b < \pi$, then the particle number is increased through the bounce $[\mu(t \to \infty) > \mu(t \to -\infty)]$; conversely, for the interval $\pi < \chi_b < 3\pi/2$ it is decreased. The intermediate case $\chi_b = \pi$ yields a time-symmetric configuration (in the limit $\mu_b \to \infty$).

Summarizing, the bounce solution is a transient configuration which in general changes the particle number and joins an asymptotic contractive phase in the past to an expansive one in the distant future, where both asymptotic states are simultaneously solutions of the standard cosmological model. Since the bounce radius is restricted to the magnitude of the Compton length ($m^{-1}$), the bounce appears as a typical quantum effect. Moreover, the thermodynamic coefficient $\beta$ surpasses the radiation-dominance value 1/3 for $r_b \to 0$ [cf. (28)] and thus signals the essential thermodynamic difference of both types of solutions.

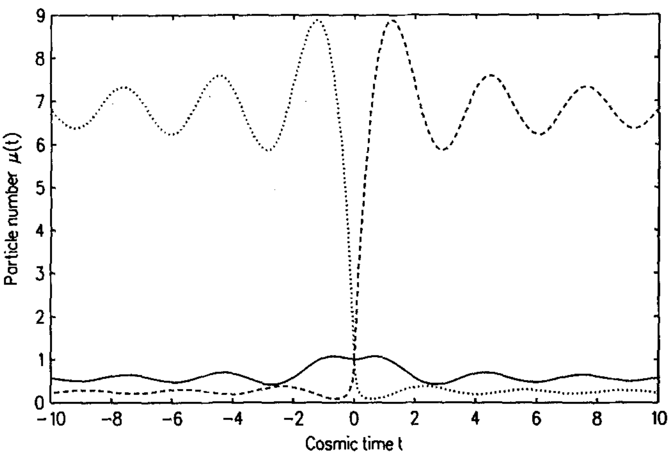


**Fig. 6.** Bounce-induced change of particle number. If the bounce is not symmetric ($\chi_b = \pi$, solid line), the ultimate particle number $\mu_*$ is changed. $\chi_b = 0.55\pi \rightsquigarrow$ particle generation (---), $\chi_b = 1.45 \rightsquigarrow$ particle annihilation ($\cdot\cdot\cdot$) ($\Lambda = \mu_b = 1$ always).

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